DISTRIBUTION OF ZEROS OF RANDOM SECTIONS OF POSITIVE LINE BUNDLES

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The goal of these notes is to give a proof of a theorem of Shiffman–Zelditch [14] about the distribution of zeros of random sections of high tensor powers of positive line bundles. Dinh–Sibony [5] gave a precise estimate of the speed of convergence. Viktoria Schmidt [13] extended recently some of these results to non-compact manifolds, in particular to zeros of modular forms.

The asymptotic properties of the zeros of random real polynomials were studied by Kac [10] in 1949; a few years later, Hammersley [8] investigated the zeros of the complexification of the Kac ensembles. While the zeros of the Hammersley ensembles tend to accumulate on the unit circle in $\mathbb{C}$, the distribution of zeros is uniform (with respect to the Fubini-Study measure on projective space) for the $SU(2)$ polynomials. Mathematical physicists [1, 2, 9, 12] study random polynomials (and more general sections) as a model for eigenfunctions of quantum chaotic maps. The large p limit of powers L^p of a positive line bundle arises in physics as the semi-classical limit in the geometric quantization of a compact Kähler manifold.

1. Projective space, Fubini Study metric

1.1. **Projective space.** Let V be a m-dimensional complex vector space and let V^* be its dual. The projective space over V is defined by $\mathbb{P}(V) := V \setminus \{0\}/\sim$, where $u \sim v$ if and only if there is $\lambda \in \mathbb{C}^*$ such that $u = \lambda\, v$. There are two other useful descriptions of the projective space. Let us denote by $G_d(V)$ the set of d-dimensional linear subspaces of V. We have canonical bijections

$$\mathbb{P}(V) \cong G_1(V) \cong G_{m-1}(V^*)$$

that is,

$*$ $\mathbb{P}(V)$ is identified to the set of complex lines through the origin in V and

$*$ A line $l \subset V$ is identified to the hyperplane $\{f \in V^* : f(u) = 0 \text{ for all } u \in l\} \subset V^*$.

For $v \in V \smallsetminus \{0\}$, we denote by $[v]$ the complex line through v and also the corresponding point $[v] \in \mathbb{P}(V)$.

We endow V with the topology induced by any norm (all norms on a finite dimensional space are equivalent). Then $\mathbb{P}(V)$ carries the quotient topology and is a Hausdorff space.

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If we choose a basis in V we have an isomorphism $V \cong \mathbb{C}^m$ and a homeomorphism $\mathbb{P}(V) \cong \mathbb{CP}^{m-1}$ where $\mathbb{CP}^{m-1}$ is the familiar projective space of dimension $m-1$:

$$\mathbb{CP}^{m-1} = \mathbb{C}^m \setminus \{0\}/\mathbb{C}^*.$$

Let $(z_0, \dots, z_{m-1}) \in \mathbb{C}^m$ be homogeneous coordinate on $\mathbb{CP}^n$. We define coordinate charts by

$$U_i = \{[z_0, \cdots, z_{m-1}] \in \mathbb{CP}^{m-1}; z_i \neq 0\}, \quad i = 0, \cdots, m-1,$$
$$\phi_i : U_i \simeq \mathbb{C}^n, \quad \phi_i([z_0, \cdots, z_{m-1}]) = \left(\frac{z_0}{z_i}, \cdots, \frac{\widehat{z_i}}{z_i}, \cdots, \frac{z_{m-1}}{z_i}\right)$$

(A hat over a variable means that this variable is skipped.) One can check easily that $\mathbb{P}(V) \cong \mathbb{CP}^{m-1}$ is an $(m-1)$–dimensional complex manifold.

1.2. **Fubini-Study form.** We work in the following with the projective space $\mathbb{P}(V^*)$ in view to applications regarding the Kodaira map. The $(1,1)$–form $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2_{h^{V^*}}$ on $V^* \setminus \{0\}$ is invariant to the action of $\mathbb{C}^*$ and descends to Kähler form

$$\omega_{FS}([f]) := \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2_{h^{V^*}} \tag{1.1}$$

on $\mathbb{P}(V^*)$ called the *Fubini-Study form.* The associated Hermitian metric is called the *Fubini-Study metric.*

If we choose an ONB in V^* we can idendify (V^*, h^{V^*}) to $\mathbb{C}^m$ endowed with the standard Hermitian scalar product. Let $(z_0, \dots, z_{m-1}) \in \mathbb{C}^m$ be homogeneous coordinate on $\mathbb{CP}^{m-1}$. The Fubini-Study form ω_{FS} on $\mathbb{CP}^{m-1}$ is given by

$$\omega_{FS}([z]) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|z_0|^2 + \ldots + |z_{m-1}|^2)$$

In a local chart ϕ_i we have for $w \in \mathbb{C}^{m-1}$

$$\begin{aligned}\omega_{FS}(w) &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_{j=1}^{m-1}|w_j|^2\right)\\ &= \frac{\sqrt{-1}}{2\pi}\left(\frac{\sum_{j=1}^{m-1} dw_j d\overline{w}_j}{1 + \sum_{j=1}^{m-1}|w_j|^2} - \frac{\sum_{j=1}^{m-1}\overline{w}_j dw_j \wedge \sum_{k=1}^{m-1} w_k d\overline{w}_k}{(1 + \sum_{j=1}^{m-1}|w_j|^2)^2}\right).\end{aligned}$$

(This is actually $(\phi^{-1})^*\omega_{FS}$.) It is useful to calculate also the Riemannian volume form on $(\mathbb{CP}^{m-1}, \omega_{FS})$, which in this chart is given by

$$\frac{1}{(m-1)!}\omega_{FS}(z)^{m-1} = \left(\frac{\sqrt{-1}}{2\pi}\right)^{m-1}\frac{dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_{m-1} \wedge d\overline{z}_{m-1}}{(1 + \sum_{j=1}^{n}|z_j|^2)^m}.$$

1.3. **Tautological and hyperplane line bundle.** Let $\mathscr{O}(-1)$ be the *tautological (universal) line bundle* on $\mathbb{P}(V^*)$, defined by

$$\mathscr{O}(-1) = \{(h, f) \in \mathbb{P}(V^*) \times V^*, f \in h \subset V^*\}. \tag{1.2}$$

For $k \in \mathbb{Z}$, set $\mathscr{O}(k) := \mathscr{O}(-1)^{-k}$, especially, $\mathscr{O}(1)$ is the dual bundle of $\mathscr{O}(-1)$; for any $[f] \in \mathbb{P}(V^*)$ the fiber $\mathscr{O}(1)_{[f]}$ is the set of linear functionals on the line $[f] \subset V^*$. The bundle $\mathscr{O}(1)$ is called *hyperplane line bundle.*

A Hermitian metric (i.e. a Hermitian scalar product) h^V on V induces naturally a Hermitian metric h^{V^*} on V^*, thus it induces a Hermitian metric $h^{\mathscr{O}(-1)}$ on $\mathscr{O}(-1)$, which is viewed as a subbundle of the trivial bundle $\mathbb{P}(V^*) \times V^* \to \mathbb{P}(V^*)$. Let $h^{\mathscr{O}(1)}$ be the Hermitian metric on $\mathscr{O}(1)$ induced by $h^{\mathscr{O}(-1)}$.

For any $v \in V$, the linear map $V^* \ni f \to f(v) \in \mathbb{C}$ defines naturally a holomorphic section σ_v of $\mathscr{O}(1)$ on $\mathbb{P}(V^*)$: for $f \in V^* \setminus \{0\}$, $\sigma_v([f]) \in \mathscr{O}(1)_{[f]}$ is the restriction of $V^* \ni f \to f(v) \in \mathbb{C}$ to the line $[f] \subset V^*$. By the definition we have

$$|\sigma_v([f])|^2_{h^{\mathscr{O}(1)}} = |f(v)|^2/|f|^2_{h^{V^*}}. \tag{1.3}$$

For a holomorphic Hermitian line bundle (F, h^F) on a complex manifold X, we will denote by R^F the curvature associated to the holomorphic Hermitian (Chern) connection ∇^F on (F, h^F). The curvature R^F is a $(1,1)$-form on X and $\sqrt{-1}R^F$ is real. For any holomorphic local frame s of F on an open set U,

$$R^F(x) = -\partial\bar{\partial}\log|s(x)|^2_{h^F} \quad \text{on } U. \tag{1.4}$$

Applying this definition for the section $s = \sigma_v$ of $\mathscr{O}(1)$ we obtain

$$R^{\mathscr{O}(1)} = -\partial\bar{\partial}\log|\sigma_v|^2_{h^{\mathscr{O}(1)}} \quad \text{on } \{x \in \mathbb{P}(V^*), \sigma_v(x) \neq 0\}. \tag{1.5}$$

By (1.1), (1.3) and (1.5) we get

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi} R^{\mathscr{O}(1)} . \tag{1.6}$$

Line bundles whose curvature is a Kähler form are called *positive*.

2. Poincaré-Lelong formula

Let Y be an *analytic hypersurface* of X (that is, for any $x \in Y$ there exists an open neighbourhood U of x and a holomorphic function $f \in \mathscr{O}(U)$ such that $Y \cap U = \{f = 0\}$; for more informations about analytic sets see [3, 4, 7]). The *current of integration on* Y, denoted $[Y] \in \Omega'^{\,n-1,n-1}(X)$, is defined by

$$(\,[Y], \varphi) = \int_{Y_{reg}} \varphi\,, \quad \varphi \in \Omega_0^{n-1,n-1}(X)\,, \tag{2.1}$$

where $Y_{\rm reg}$ is the complex manifold of regular points of Y (endowed with the canonical orientation).

Pierre Lelong showed that this is well defined (cf. [4, III-2.6], [7, p. 32]) which amounts to showing that Y_{reg} has finite volume locally near $Y_{\rm sing}$.

Let F be a holomorphic line bundle and $s \in H^0(X, F)$ be a holomorphic section of F on X. We define the *divisor* of s by

$$\operatorname{Div}(s) = \textstyle\sum_V \operatorname{ord}_V(s)\cdot V, \tag{2.2}$$

where the sum runs formally over all irreducible analytic hypersurfaces of X and $\operatorname{ord}_V(s) \in \mathbb{Z}$ is the order of s along V. Locally there exist only a finite number of V's such that $\operatorname{ord}_V(s) \neq 0$.

If $D = \sum c_j D_j$ with $c_j \in \mathbb{Z}$ is a divisor on X, where D_j are irreducible hypersurfaces, we define the current of integration on D by

$$[D] = \sum c_j [D_j]\,. \tag{2.3}$$

Theorem 2.1 (Poincaré-Lelong formula [4, III-2.15], [3]). *Let f be a holomorphic function defined on an open set of X and which is non-zero on any connected component. Then $\log|f|$ is locally integrable and*

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2 = [\,\mathrm{Div}(f)\,]\,. \tag{2.4}$$

Let s be a holomorphic section of a Hermitian holomorphic line bundle (F, h^F) on X. Then

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s|^2_{h^F} = [\mathrm{Div}(s)] - \frac{\sqrt{-1}}{2\pi}R^{(F,h^F)}. \tag{2.5}$$

3. Convergence of the induced Fubini-Study metrics

3.1. **The asymptotic expansion of the Bergman kernel.** Let (X,ω) be a compact complex Kähler manifold and $\dim X = n$. We consider a holomorphic positive line bundle L on X, endowed with a Hermitian metric h^L such that the curvature R^L associated to h^L verifies

$$\frac{\sqrt{-1}}{2\pi}R^L = \omega\,. \tag{3.1}$$

We introduce the Riemannian metric g^{TX} on TX associated to ω, i.e. $g^{TX}(u,v) = \omega(u,Jv)$, where J denotes the complex structure of X. The Riemannian volume form on (X, g^{TX}) is $dv_X = \omega^n/n!$. The metric h^L on L induces metrics h^{L^p} on L^p.

On $\mathscr{C}^\infty(X,L^p)$ we define an L^2–scalar product $\langle\cdot,\cdot\rangle$ by setting for $s, s' \in \mathscr{C}^\infty(X,L^p)$

$$\langle s, s'\rangle = \int_X \langle s(x), s'(x)\rangle_{L^p}\, dv_X(x)\,. \tag{3.2}$$

The scalar product $\langle\cdot,\cdot\rangle$ induces an L^2-metric $h^{H^0(X,L^p)}$ on the space $H^0(X,L^p)$ of holomorphic sections of L^p on X.

Let $P_p(\cdot,\cdot)$ be the smooth kernel of the orthogonal projection P_p from $(\mathscr{C}^\infty(X,L^p),\langle\cdot,\cdot\rangle)$ onto $H^0(X,L^p)$, with respect to $dv_X(x')$. $P_p(\cdot,\cdot)$ is called the *Bergman kernel* of L^p ([11, Def. 1.4.1]). Let $d_p = \dim H^0(X,L^p)$ and $\{S_i^p\}_{i=1}^{d_p}$ be an orthonormal basis of $(H^0(X,L^p), h^{H^0(X,L^p)})$, then

$$P_p(x,x) = \sum_{i=1}^{d_p} |S_i^p(x)|^2_{h^{L^p}}. \tag{3.3}$$

Theorem 3.1 (Zelditch, Catlin, [11, Th. 4.1.1]). *Assume that (3.1) holds. There exist coefficients $\boldsymbol{b}_r \in \mathscr{C}^\infty(X)$ which are polynomials in R^{TX} and R^L and their derivatives such that*

$$\boldsymbol{b}_0 = 1, \tag{3.4}$$

and for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$,

$$\Big|P_p(x,x) - \sum_{r=0}^{k}\boldsymbol{b}_r(x)p^{n-r}\Big|_{\mathscr{C}^l(X)} \leqslant C_{k,\,l}\,p^{n-k-1}. \tag{3.5}$$

In particular we have uniformly on X

$$P_p(x,x) = p^n + \mathscr{O}(p^{n-1}) \tag{3.6}$$

Example 3.2 (Bergman kernel of $\mathscr{O}(p)$). For any $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, the map $\mathbb{C}^{n+1} \ni z \to \prod_{j=0}^n z_j^{\alpha_j}$ is naturally identified to a holomorphic section of $\mathscr{O}(|\alpha|)$ on $\mathbb{CP}^n$, denoted by s_α. Verify that for $\alpha, \beta \in \mathbb{N}^{n+1}$, $|\alpha| = |\beta|$,

$$\langle s_\alpha, s_\beta \rangle = \frac{\alpha!}{(n+|\alpha|)!}\, \delta_{\alpha\,\beta}\,,$$

and $\{s_\alpha : \alpha \in \mathbb{N}^{n+1}\,,\, |\alpha| = p\}$ is an ONB of $H^0(\mathbb{CP}^n, \mathscr{O}(p))$. Thus we get the Bergman kernel $P_p(z,w)$ associated to $\mathscr{O}(p)$ is

$$P_p(z,w) = \sum_{\alpha \in \mathbb{N}^{n+1},\, |\alpha| = p} \frac{(n+p)!}{\alpha!}\, s_\alpha(z) \otimes s_\alpha(w)^*.$$

Especially for any $z \in \mathbb{CP}^n$, we have

$$P_p(z,z) = (n+p)!/p! = p^n + b_1 p^{n-1} + \ldots + b_n \in \mathbb{C}[p].$$

3.2. **The Kodaira map and Tian's theorem.** Let $x \in X$. The evaluation map $H^0(X, L^p) \to L^p_x$, $s \mapsto s(x)$, is a linear map whose kernel is either a hyperplane or the the whole space. Note that by (3.6) and the compactness of X there exists p_0 such that for all $p \geqslant p_0$ and all $x \in X$ we have $P_p(x,x) \neq 0$. Especially for every $x \in X$ there exists $s \in H^0(X, L^p)$ such that $s(x) \neq 0$. The *Kodaira map* of L^p is defined for $p \geqslant p_0$ by

$$\begin{aligned} &\Phi_p : X \longrightarrow \mathbb{P}(H^0(X, L^p)^*), \\ &\Phi_p(x) = \{s \in H^0(X, L^p) : s(x) = 0\}\,. \end{aligned} \tag{3.7}$$

The Kodaira map defines a canonical isomorphism from $\Phi_p^* \mathscr{O}(1)$ to L^p on X, and under this isomorphism, we have [11, Th. 5.1.3]:

$$h^{\Phi_p^* \mathscr{O}(1)}(x) = P_p(x,x)^{-1} h^{L^p}(x). \tag{3.8}$$

Here $h^{\Phi_p^* \mathscr{O}(1)}$ is the metric on $\Phi_p^* \mathscr{O}(1)$ induced by the metric $h^{\mathscr{O}(1)}$ on $\mathscr{O}(1) \to \mathbb{P}(H^0(X, L^p)^*)$.

Theorem 3.3 (Tian-Ruan). *Assume that* (3.1) *holds. Then the induced Fubini–Study form* $\frac{1}{p}\Phi_p^*(\omega_{FS})$ *converges in the* $\mathscr{C}^\infty$ *topology to* ω *; for any* $l \geqslant 0$ *there exists* $C_l > 0$ *such that*

$$\left| \frac{1}{p}\, \Phi_p^*(\omega_{FS}) - \omega \right|_{\mathscr{C}^l(X)} \leqslant \frac{C_l}{p^2}. \tag{3.9}$$

Proof. From (1.4), (1.5) and (3.8), we get

$$\frac{1}{p} \Phi_p^*(\omega_{FS}) - \frac{\sqrt{-1}}{2\pi} R^L = \frac{\sqrt{-1}}{2\pi p}\, \partial\overline{\partial} \log P_p(x,x)\,. \tag{3.10}$$

By (3.6),

$$\partial\overline{\partial} \log P_p(x,x) = \partial\overline{\partial} \log(\boldsymbol{b}_0(x)) + \mathscr{O}(1/p) = \mathscr{O}(1/p). \tag{3.11}$$

From (3.10) and (3.11), we get (3.9). □

4. Distribution of zeros of random sections

As an application of Theorem 3.3, we study the distribution of zeros of random sections. Let dS be the usual volume form on the $2m-1$-dimensional unit sphere $S^{2m-1}:=\{\lambda\in\mathbb{C}^m; |\lambda|=1\}$, then

$$\mathrm{vol}(S^{2m-1})=\int_{S^{2m-1}}dS=\frac{2\,\pi^m}{(m-1)!}. \tag{4.1}$$

Let $h^{H^0(X,L^p)}$ be the L^2-metric on $H^0(X,L^p)$ induced by g^{TX},h^L. Let $d\mu_p$ be the equidistribution probability measure on the unit sphere

$$SH^0(X,L^p):=\{s\in H^0(X,L^p); |s|_{h^{H^0(X,L^p)}}=1\}. \tag{4.2}$$

Recall that $d_p=\dim H^0(X,L^p)$. We fix an ONB $\{S_i^p\}_{i=1}^{d_p}$ of $(H^0(X,L^p),h^{H^0(X,L^p)})$, then we can identify S^{2d_p-1} to $SH^0(X,L^p)$ by $S^{2d_p-1}\ni(z_1,\cdots,z_{d_p})\to\sum_i z_iS_i^p$, and we have

$$d\mu_p=\frac{dS}{\mathrm{vol}(S^{2d_p-1})}. \tag{4.3}$$

As in (2.3), we denote by $[\mathrm{Div}(s)]$ the current of integration on the divisor $\mathrm{Div}(s)$ of the section $s\in H^0(X,L^p)$. We view $[\mathrm{Div}(s)]$ as a $\Omega'^{1,1}(X)$-valued Random variable as s varies over the probability space $(SH^0(X,L^p),d\mu_p)$. The *expected value of* $[\mathrm{Div}(s)]$ is the current $E([\mathrm{Div}(s)])\in\Omega'^{1,1}(X)$ defined by

$$E([\mathrm{Div}(s)])(\varphi)=\int_{S^{2d_p-1}}\Big(\Big[\mathrm{Div}\Big(\sum_{i=1}^{d_p}\lambda_iS_i^p\Big)\Big],\varphi\Big)d\mu_p(\lambda)\,,\quad \varphi\in\Omega^{n-1,n-1}(X). \tag{4.4}$$

The *expected distribution of zeros* of the random section $s\in H^0(X,L^p)$ is by definition the expectation of the normalized zero divisor $\frac{1}{p}\mathrm{Div}(s)$.

Theorem 4.1. *Let $p\in\mathbb{N}$ be large enough so that the Kodaira map Φ_p is defined on X. Then expected distribution of zeros of the random section $s\in H^0(X,L^p)$ is given by*

$$E([\mathrm{Div}(s_p)])=\Phi_p^*(\omega_{\mathrm{FS}})\,. \tag{4.5}$$

Proof. By Poincaré-Lelong formula, Theorem 2.1, for $s\in H^0(X,L^p)$,

$$[\mathrm{Div}(s)]=\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s|^2_{h^{L^p}}+p\,\omega. \tag{4.6}$$

Thus the current $\frac{1}{p}[\mathrm{Div}(s)]$ is a representative of $c_1(L)$ for any $s\in H^0(X,L^p)$. Let $\varphi\in\Omega^{n-1,n-1}(X)$. By (3.10), (4.6), for $s\in H^0(X,L^p)$,

$$\begin{aligned}([\mathrm{Div}(s)],\varphi)&=\int_X\Big(\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s|^2_{h^{L^p}}+p\,\omega\Big)\wedge\varphi\\ &=\int_X\Phi_p^*(\omega_{\mathrm{FS}})\wedge\varphi+\frac{\sqrt{-1}}{2\pi}\int_X\partial\overline{\partial}\log|P_p(x,x)^{-1/2}s(x)|^2_{h^{L^p}}\wedge\varphi\\ &=\int_X\Phi_p^*(\omega_{\mathrm{FS}})\wedge\varphi+\frac{\sqrt{-1}}{2\pi}\int_X\log|P_p(x,x)^{-1/2}s(x)|^2_{h^{L^p}}\,\partial\overline{\partial}\varphi.\end{aligned} \tag{4.7}$$

Let $e_L(x)$ be an unit vector of L at $x \in X$. We define local holomorphic functions f_i^p by $S_i^p = f_i^p e_L^{\otimes p}$. By (3.3),

$$\psi(x) := P_p(x,x)^{-1/2}(f_1^p(x), \dots, f_{d_p}^p(x)) \in S^{2d_p-1}.$$

By a change of the unit vector $e_L(x)$, $\psi(x)$ changes by multiplication with a unit complex number.
For $u = (u_1, \cdots, u_{d_p})$, $v = (v_1, \cdots, v_{d_p}) \in \mathbb{C}^{d_p}$, we denote by

$$u \cdot v = \sum_{i=1}^{d_p} u_i v_i\,, \qquad u \cdot S^p = \sum_{i=1}^{d_p} u_i S_i^p\,. \tag{4.8}$$

Observe that for the function $S^{2d_p-1} \ni u \longmapsto \int_{S^{2d_p-1}} \log|\lambda \cdot u|^2 d\mu_p(\lambda)$ is constant and equal to, say, c_p. Thus

$$x \mapsto \int_{S^{2d_p-1}} \log\left|P_p(x,x)^{-1/2}\lambda\cdot S^p\right|^2_{h^{L^p}} d\mu_p(\lambda) = \int_{S^{2d_p-1}} \log|\lambda\cdot\psi(x)|^2 d\mu_p(\lambda) = c_p \tag{4.9}$$

is a constant function on X. Thus from (4.4), (4.7) and (4.9), we get

$$\begin{aligned} E([\mathrm{Div}(s)])(\varphi) &= \int_{S^{2d_p-1}} ([\mathrm{Div}(\lambda\cdot S^p)]\,,\varphi)\, d\mu_p(\lambda) \\ &= \int_X \Phi_p^*(\omega_{\mathrm{FS}})\wedge\varphi + \frac{\sqrt{-1}}{2\pi}\int_X c_p\,\partial\bar\partial\varphi = \int_X \Phi_p^*(\omega_{\mathrm{FS}})\wedge\varphi. \end{aligned} \tag{4.10}$$

The proof of Theorem 4.1 is complete. □

Lemma 4.2. *For $u, v \in S^{2d_p-1}$, set*

$$A_p(u,v) = \int_{S^{2d_p-1}} \log|\lambda\cdot u|\,\log|\lambda\cdot v| d\mu_p(\lambda). \tag{4.11}$$

Then there exists a constant C_p (depending only on p) such that $A_p(u,v) - C_p$ is uniformly bounded for $(u,v) \in S^{2d_p-1}\times S^{2d_p-1}$, $p \in \mathbb{N}$.

Proof. Set $|Z|^2 = \sum_{i=1}^m |z_i|^2$, and dZ the standard volume form on $\mathbb{C}^m$ for $m \in \mathbb{N}$. We consider the Gaussian integral

$$\widetilde{A}_p(u,v) := \int_{\mathbb{C}^{d_p}} e^{-|Z|^2}\log|z\cdot u|\,\log|z\cdot v|\,dZ. \tag{4.12}$$

We evaluate (4.12) in two ways. First we choose the coordinates in $\mathbb{C}^{d_p}$ such that $u = (1,0,\cdots,0)$, $v = (v_1, v_2, 0, \cdots, 0)$. Set $z' = (z_1, z_2)$, $\tilde z = (z_3, \cdots, z_{d_p})$, $v' = (v_1, v_2)$, and with c_p in (4.9),

$$\begin{aligned} \rho(v') &= \int_{\mathbb{C}^2} e^{-|Z'|^2}\log|z_1|\,\log|z_1v_1 + z_2v_2|\,dZ', \\ C'_p &= \int_0^\infty e^{-r^2}r^{2d_p-1}\Big[\mathrm{vol}(S^{2d_p-1})(\log r)^2 + c_p\log r\Big]\,dr. \end{aligned} \tag{4.13}$$

Then from (4.12), we have

$$\widetilde{A}_p(u,v) = \Big[\int_{\mathbb{C}^{d_p-2}} e^{-|\widetilde Z|^2}d\widetilde Z\Big]\rho(v') = \pi^{d_p-2}\rho(v'). \tag{4.14}$$

By the Cauchy-Schwarz inequality, for $v' \in S^3$, we have

$$\begin{aligned}(4.15)\quad |\rho(v')| &\leqslant \Big[\int_{\mathbb{C}^2} e^{-|Z'|^2}(\log|z_1|)^2 dZ'\Big]^{1/2}\Big[\int_{\mathbb{C}^2} e^{-|Z'|^2}\Big(\log|z_1v_1+z_2v_2|\Big)^2 dZ'\Big]^{1/2}\\ &= \int_{\mathbb{C}^2} e^{-|Z'|^2}(\log|z_1|)^2 dZ' < +\infty.\end{aligned}$$

Now we use spherical coordinates $Z = r\lambda$ with $\lambda \in S^{2d_p-1}$. We have

$$\widetilde{A}_p(u,v) = \int_0^\infty dr \int_{S^{2d_p-1}} e^{-r^2} r^{2d_p-1}(\log r + \log|\lambda\cdot u|)(\log r + \log|\lambda\cdot v|)dS(\lambda).$$

By using (4.1), (4.3), (4.9), (4.13) and the above equation, we have

$$\begin{aligned}(4.16)\quad \widetilde{A}_p(u,v) &= C'_p + \int_0^\infty e^{-r^2} r^{2d_p-1} dr \int_{S^{2d_p-1}} \log|\lambda\cdot u|\,\log|\lambda\cdot v|\,dS(\lambda)\\ &= C'_p + \pi^{d_p}\int_{S^{2d_p-1}} \log|\lambda\cdot u|\,\log|\lambda\cdot v|\,d\mu_p(\lambda).\end{aligned}$$

From (4.14), (4.16), we get

$$(4.17)\quad A_p(u,v) = \pi^{-d_p}(\widetilde{A}_p(u,v) - C'_p) = \pi^{-2}\rho(v') - C'_p/\pi^{d_p}.$$

The proof of Lemma 4.2 is completed. □

Now we consider the probability space $\Omega = \prod_{p=1}^{\infty} SH^0(X, L^p)$ with the probability measure $\mu = \prod_{p=1}^{\infty}\mu_p$. We denote by $\mathrm{s} = \{s_p\} \in \Omega$.

The main result is the following theorem which states that for a random variable $\mathrm{s} = \{s_p\} \in \Omega$, the sequence of zeros of the sections s_p are asymptotically uniformly distributed.

Theorem 4.3 (Shiffman-Zelditch [14], Dinh-Sibony [5], [11, Th. 5.3.3]). *For μ-almost all $\mathrm{s} = \{s_p\} \in \Omega$, $\frac{1}{p}[\mathrm{Div}(s_p)] \to \omega$ weakly in the sense of distributions, i.e. for all continuous $(n-1, n-1)$-forms φ, we have*

$$(4.18)\quad \lim_{p\to\infty}\frac{1}{p}\Big([\mathrm{Div}(s_p)], \varphi\Big) = \int_X \omega\wedge\varphi.$$

Proof. At first, from (4.6),

$$(4.19)\quad \frac{1}{p}\,|([\mathrm{Div}(s_p)], \varphi)| \leqslant \frac{1}{p}\Big([\mathrm{Div}(s_p)], \omega^{n-1}\Big)\,|\varphi|_{\mathscr{C}^0} = |\varphi|_{\mathscr{C}^0}\int_X c_1(L)^n.$$

By considering a countable $\mathscr{C}^0$-dense family of φ, we need only to consider one φ. Consider the random variables

$$(4.20)\quad Y_p(\mathrm{s}) = \Big(\frac{1}{p}[\mathrm{Div}(s_p)] - \frac{1}{p}\Phi_p^*(\omega_{\mathrm{FS}}), \varphi\Big).$$

By Theorem 3.3, we need to prove that μ-almost surely for $\mathrm{s} \in \Omega$, as $p \to \infty$,

$$(4.21)\quad Y_p(\mathrm{s}) \to 0.$$

By Theorem 4.1, we get

$$(4.22)\quad E\Big(|Y_p(\mathrm{s})|^2\Big) = \frac{1}{p^2}E\Big(|([\mathrm{Div}(s_p)], \varphi)|^2\Big) - \frac{1}{p^2}\Big|(\Phi_p^*(\omega_{\mathrm{FS}}), \varphi)\Big|^2.$$

By (4.4), (4.7) and (4.9),

$$E\left(\left|([\mathrm{Div}(s_p)],\varphi)\right|^2\right)=\left|(\Phi_p^*(\omega_{\mathrm{FS}}),\varphi)\right|^2 + \frac{1}{4\pi^2}\int_X\int_X(\partial\bar{\partial}\varphi(x))(\overline{\partial\bar{\partial}\varphi(y)})\int_{S^{2d_p-1}}\log|P_p(x,x)^{-1/2}\lambda\cdot S^p(x)|^2_{h^{L^p}} \times \log|P_p(y,y)^{-1/2}\lambda\cdot S^p(y)|^2_{h^{L^p}}d\mu_p(\lambda). \tag{4.23}$$

From Lemma 4.2, (4.22) and (4.23), we get

$$E\left(|Y_p(\mathrm{s})|^2\right)=\frac{1}{4\pi^2p^2}\int_X\int_X(\partial\bar{\partial}\varphi(x))(\overline{\partial\bar{\partial}\varphi(y)})A_p(\psi(x),\psi(y))d\mu_p(\lambda)=O(p^{-2}). \tag{4.24}$$

Thus

$$\int_\Omega\sum_{p=1}^\infty|Y_p(\mathrm{s})|^2d\mu(\mathrm{s})=\sum_{p=1}^\infty\int_\Omega|Y_p(\mathrm{s})|^2d\mu(\mathrm{s})=\sum_{p=1}^\infty E\left(|Y_p(\mathrm{s})|^2\right)<+\infty. \tag{4.25}$$

Hence μ-almost surely, $Y_p\to 0$ as $p\to\infty$. □

Dinh-Sibony [5] obtain sharp results on the limit distribution of common zeros using the formalism of meromorphic transforms. Namely they prove the following result about the *speed of convergence*:

Theorem 4.4. *Let $\varepsilon>0$ be given. There exist constants $c>0$, $m\geqslant 0$, $\alpha>0$ such that for all p large enough and all $(n-1,n-1)$ test forms φ of class $\mathscr{C}^2$ we have*

$$\mu_p\Big\{s\in SH^0(X,L^p)\,:\,\Big\langle\tfrac{1}{p}[\mathrm{Div}(s)]-\omega,\varphi\Big\rangle\geqslant\varepsilon\Big\}\leqslant c\|\varphi\|_{\mathscr{C}^2}\,p^{mn}\,\exp(-\varepsilon\alpha p)\,. \tag{4.26}$$

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