Interacting particle systems out of equilibrium

Thomas Kriecherbauer
Fakultät für Mathematik, Ruhr-Universität Bochum

Joachim Krug
Institut für Theoretische Physik, Universität zu Köln
E-mail: thomas.kriecherbauer@ruhr-uni-bochum.de and krug@thp.uni-koeln.de

Abstract. These notes are based on lectures delivered by the authors within the framework of SFB/TR12 Symmetries and universality in mesoscopic systems at Langeoog in November 2007.

1. Introduction and outline

The purpose of these lecture notes is twofold. First, we explain in general terms what physicists mean by the distinction between equilibrium and nonequilibrium systems, and describe different types of nonequilibrium behavior. Second, we introduce a class of stochastic models, known as interacting particle systems, which provide useful models for various nonequilibrium phenomena. Technically, interacting particle systems are (discrete or continuous time) Markov chains, and we will argue below that a simple yet precise criterion for the equilibrium vs. nonequilibrium character of a given system can be formulated within the general theory of Markov chains.

The focus of the lectures is on the asymmetric exclusion process (ASEP) as a paradigmatic model for driven transport of a single conserved quantity. After reviewing some elementary properties of the ASEP, we describe a conjecture due to Kardar, Parisi and Zhang (KPZ) which claims that the fluctuation properties of a large class of models similar to the ASEP display universal features.

The second part of these notes is devoted to explaining a result of Johansson [13] that can be applied to discrete TASEP with step initial conditions yielding precise information on the flux of particles. This result is one example of many from the class of exactly solvable combinatorial models which display random matrix statistics. Other models of such type are: Longest increasing subsequence of random permutations, polynuclear growth model of Prähofer and Spohn, last passage percolation...
with geometric or exponential waiting times, random domino tilings of the Aztec
diamond resp. random tilings of hexagons with rhombi.

The analysis of all these models naturally falls into two parts. One is the
combinatorial aspect which is connected to Young Tableaux and the representation
theory of the permutation group. The second aspect is the asymptotic analysis of the
resulting formula which often is related to the method of orthogonal polynomials that
has been introduced in the theory of Random Matrices. We attempt to explain both
aspects with the example at hand.

2. Equilibrium and nonequilibrium states

The most fundamental concept of statistical physics is the distinction between
microstates and macrostates in the description of systems with many degrees of freedom.
To fix ideas, consider a classical $N$-particle system described by a Hamilton function
$H(q, p)$ of position variables $q = (q_1, ..., q_{dN})$ and momenta $p = (p_1, ..., p_{dN})$. Particles
move in a region $Ω ⊂ \mathbb{R}^d$ of volume $V = |Ω|$. Then a microstate is simply a point $(q, p)$
in phase space, whereas a macrostate will be defined for the purposes of these lectures as
a measure $P_X(q, p) dq dp$ parameterized by a set of macroscopic state variables $X$. Here
$P_X(q, p)$ is a function on phase space and $dq dp$ denotes the canonical Liouville measure.

In this perspective, equilibrium states are a subclass of macrostates which are
attained at long times by a system that is isolated or in contact with a time-independent,
spatially uniform environment. Characteristic properties of equilibrium states are that
• macroscopic observables are time-independent and spatially homogeneous, and
• there are no macroscopic currents (e.g., of mass or energy).

The two most important examples of equilibrium states are the following:

a.) In an isolated system the energy $E$ is conserved, the appropriate macrovariables are
$X = (E, V, N)$ and the equilibrium state is the uniform measure on the energy shell
$\{(q, p) : H(q, p) = E\}$. This is known in physics as the microcanonical measure.

b.) In a system at constant temperature $T$ particles exchange energy with the walls
of the box $Ω$ in such a way that the mean energy is fixed. The appropriate
macrovariables are then $X = (T, V, N)$ and the equilibrium state is of the form

$$P_{T,V,N} \sim \exp[-\beta H], \quad \beta = 1/T,$$

known as the canonical measure.

Having defined equilibrium states, we may say that nonequilibrium states arise whenever
the conditions for the establishment of equilibrium are not fulfilled. As such, this
definition is about as useful as it would be to define some area of biology as the study
of non-elephants. We can be somewhat more precise by making a distinction between
(i) **Systems approaching equilibrium.** By definition, the macrostate of such a system is time-dependent. In addition, systems in this class often become spatially inhomogeneous; an important and much studied case are systems undergoing phase separation [3].

(ii) **Nonequilibrium stationary states (NESS).** These systems are kept out of equilibrium by external influences. They are stationary, in the sense that macroscopic state variables are time-independent, and they may or may not be spatially homogeneous. In any case they are characterized by non-vanishing macroscopic currents.

Examples for NESS are

- **Heat conduction.** In a system with boundaries held at different temperatures there is a stationary energy current proportional to the temperature gradient (Fourier’s law).
- **Diffusion.** In a system coupled to particle reservoirs held at different densities there is a mass current proportional to the density gradient (Fick’s law).
- **Electric conduction.** Here particles are charged and move under the influence of a constant electric field. The particle current is proportional to the field strength (Ohm’s law).

Among these three examples, the first two can be further characterized as boundary driven, in the sense that the NESS is maintained by boundary conditions on the quantity that is being transported (heat, mass), whereas the last example illustrates a bulk-driven NESS.

NESS are the simplest examples of nonequilibrium states. Nevertheless, their description in the framework of classical Hamiltonian mechanics is conceptually subtle and technically demanding (see, e.g., [40]). The main reason is that a Hamiltonian system under constant driving inevitably accumulates energy. In order to allow for the establishment of a steady state, dissipation has to be introduced through the coupling to an external reservoir, that is, a system with an infinite number of degrees of freedom.

These difficulties can be avoided by starting from stochastic microscopic dynamics. While less realistic on the microscopic level, stochastic models provide a versatile framework for addressing fundamental questions associated with the behavior of many-particle systems far from equilibrium. The class of models of interest here are known in the probabilistic community as interacting particle systems. These are lattice models with a discrete (finite or infinite) set of states associated with each lattice site and local interactions. We focus specifically on exclusion processes, which are introduced in the next section.
3. An introduction to exclusion processes

3.1. Definition

The simple exclusion process was introduced in 1970 by Frank Spitzer [34]. Particles occupy the sites of a $d$-dimensional lattice, which for the purposes of this discussion will be taken to be a finite subset $\Omega \subset \mathbb{Z}^d$. The particles are indistinguishable, which implies that a microstate or configuration of the system is given by

$$\eta = \{\eta_x\}_{x \in \Omega} \in \{0, 1\}^\Omega,$$

where $\eta_x = 0$ ($1$) if site $x$ is vacant (occupied). The dynamics can be informally described as follows (for a detailed construction see [34, 24]):

- Each particle carries a clock which rings according to a Poisson process with unit rate (i.e., the waiting times between rings are exponentially distributed).
- When the clock rings the particle selects a direction $k$ with probability $q_k$, $k = 1, \ldots, 2d$, and attempts to jump to the corresponding nearest neighbor site; the set $\{q_k\}$ of probabilities define the directional bias in the motion of the particles.
- The jump is performed if the target site is vacant and discarded otherwise; this step implements the exclusion interaction between particles and enforces the single occupancy constraint $\eta_x = 0$ or $1$.

Together these rules define the exclusion process as a continuous time Markov chain on a finite state space; some general properties of such chains will be discussed in the next section. Inhomogeneity associated with sites or particles can be introduced into the model at the level of the waiting times and/or at the level of the bias probabilities $\{q_k\}$, see [20].

We next restrict the discussion to the one-dimensional case. Then $\{q_k\} = \{q_R, q_L\} = \{q, 1-q\}$ where $q_R = q$ ($q_L = 1-q$) is the probability to jump to the right (left). The following cases are of interest:

(i) $q = 1/2$ defines the symmetric simple exclusion process (SSEP). We will see below that this is really an equilibrium system. However, when defined on a finite lattice of sites $x = 1, \ldots, L$ and supplemented with boundary rates $\alpha, \beta, \gamma, \delta$ which govern the injection $(\alpha, \delta)$ and extraction $(\gamma, \beta)$ of particles at the boundary sites $i = 1$ and $i = L$, this model provides a nontrivial example for a boundary-driven NESS [5].

(ii) $q \neq 1/2$ defines the asymmetric simple exclusion process (ASEP). When considered on the one-dimensional ring (a lattice with periodic boundary conditions) the system attains a bulk-driven NESS in which there is a non-vanishing stationary mass current. This is the simplest realization of a driven diffusive system [30].
Note that the boundary conditions are crucial here. On a finite lattice with closed ends, which prevent particles from entering or leaving the system, an equilibrium state is established in which the bias in the jump probability is compensated by a density gradient; this is the discrete analog of a gas in a gravitational field, as described by the barometric formula. A third possibility is to consider a finite lattice with open ends at which particles are injected and extracted at specified rates [18]. This leads to a NESS with a surprisingly complex structure, see [2] for review.

(iii) \( q = 1 \) or 0 defines the \textit{totally asymmetric simple exclusion process} (TASEP). In contrast to the case of general \( q \), this process can also be formulated in discrete time [42]: In one time step \( t \to t + 1 \), all particles attempt to move to the right (say) simultaneously and independently with probability \( \pi \in (0, 1] \); moves to vacant sites are accepted and moves to occupied sites discarded. For \( \pi \to 0 \) the process reduces to the continuous time TASEP in rescaled time \( \pi t \), while for \( \pi = 1 \) it becomes a deterministic cellular automaton which has number 184 in Wolfram’s classification [41]. The case of general \( \pi \) has been studied mostly in the context of vehicular traffic modeling [31, 4].

Note that in terms of the waiting time picture sketched above, the discrete time dynamics corresponds to replacing the exponential waiting time distribution by a geometric distribution with support on integer times only. The exponential and geometric waiting time distributions are the only ones that encode \textit{Markovian} dynamics [21]. The waiting time representation will play an important role in the exact solution of the discrete time TASEP presented below.

3.2. \textit{Continuous time Markov chains}

Before discussing some specific properties of exclusion processes, we outline the general setting of continuous time Markov chains (see [26] for an introduction). Consider a Markov chain with a finite number of states \( i = 1, ..., C \) and transition rates \( \Gamma_{ij} \). The rates define the dynamics in the following way:

When the chain is in state \( i \) at time \( t \), a transition to state \( j \) occurs in the time interval \([t, t + dt]\) with probability \( \Gamma_{ij}dt \).

The key quantity of interest is the transition probability

\[
P_{ki}(t) = \text{Prob}[\text{state } i \text{ at } t| \text{state } k \text{ at } 0] \equiv P_i(t)
\]

where the initial state \( k \) is included through the initial condition \( P_i(0) = \delta_{ik} \). The transition probability satisfies the evolution equation

\[
\frac{d}{dt}P_i = \sum_j \Gamma_{ji}P_j - \sum_j \Gamma_{ij}P_i = \sum_j A_{ji}P_j,
\]  

(1)

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which is known as the master equation in physics [39] and as the forward equation in the theory of stochastic processes [26]. Here the generator matrix

\[ A_{ij} = \begin{cases} \Gamma_{ij} : i \neq j \\ -\sum_k \Gamma_{ik} : i = j \end{cases} \]

has been introduced. The master equation simply accounts for the balance of probability currents going in and out of each state of the Markov chain. To bring out this structure we rewrite (1) in the form

\[ \frac{d}{dt} P_i = \sum_j K_{ij}, \quad K_{ij} = \Gamma_{ji} P_j - \Gamma_{ij} P_i, \quad (2) \]

where \( K_{ij} \) is the net probability current between states \( i \) and \( j \) [43]. If the chain is irreducible, in the sense that every state can be reached from every other state through a connected path of nonzero transition rates, the solution of (1) approaches at long times a unique, stationary invariant measure \( P^*_i \) determined by the condition

\[ \sum_j A_{ji} P^*_j = 0. \quad (3) \]

The invariant measure is the left eigenvector of the generator matrix, with eigenvalue zero. Based on (2) we can rewrite (3) as

\[ \sum_j K^*_{ji} = 0 \quad \text{with} \quad K^*_{ji} = \Gamma_{ji} P^*_j - \Gamma_{ij} P^*_i. \quad (4) \]

Two classes of Markov chains may now be distinguished depending on how the stationarity condition (4) is realized:

(i) \( K^*_{ij} = 0 \ \forall \ i, j \). In this case the probability currents cancel between any two states \( i, j \),

\[ \Gamma_{ij} P^*_i = \Gamma_{ji} P^*_j, \quad (5) \]

a condition that is known in physics as detailed balance. In the mathematical literature Markov chains with this property are called reversible, because (5) implies that the time-reversed process is identical in distribution to the original process [26, 15]. Detailed balance or reversibility is a fundamental property that any stochastic model of a physical system in equilibrium must satisfy, because equilibrium states are distinguished by invariance under time reversal.

(ii) \( K^*_{ij} \neq 0 \) at least for some pairs of states \( i, j \). Such a Markov chain is irreversible and describes a system in a NESS.

Examples for both kinds of situations will be described in the next section.
3.3. Stationary measure of the exclusion process

We consider the ASEP on a ring of \( L \) sites with a fixed number \( N \) of particles. The total number of microstates \( \eta \) is then \( C = \binom{L}{N} \) and the transition rates are

\[
\Gamma(\eta \rightarrow \eta') = \begin{cases} 
q & : (\ldots \bullet \ldots) \rightarrow (\ldots \circ \ldots) \\
1 - q & : (\ldots \circ \ldots) \rightarrow (\ldots \bullet \ldots) \\
0 & : \text{else.}
\end{cases}
\]

(6)

Here \((\ldots \bullet \ldots)\) denotes a local configuration with an occupied site \((\bullet)\) to the left of a vacant site \((\circ)\), and it is understood that only configurations \(\eta, \eta'\) that differ by the exchange of a single particle-vacancy pair are connected through nonzero transition rates. The stationary measure \(P^*(\eta)\) is determined by the condition

\[
\sum_{\eta'} \Gamma(\eta' \rightarrow \eta)P^*(\eta') = \sum_{\eta'} \Gamma(\eta \rightarrow \eta')P^*(\eta) \quad \forall \eta.
\]

(7)

As an educated guess, let us assume that the invariant measure is simply uniform on the state space,

\[
P^*(\eta) = \left(\binom{L}{N}\right)^{-1} \Rightarrow K^*(\eta, \eta') = [\Gamma(\eta' \rightarrow \eta) - \Gamma(\eta \rightarrow \eta')]\left(\binom{L}{N}\right)^{-1}.
\]

(8)

We discuss separately the symmetric and the asymmetric process.

- \(q = 1/2\) (SSEP). Here the rate \(q = 1 - q = 1/2\) for all allowed processes, and for each allowed process the reverse process occurs at the same rate. We conclude that detailed balance holds in this case, \(K^* = 0\), and the SSEP is reversible as announced previously.

- \(q \neq 1/2\) (ASEP). Because for any allowed process with rate \(q\) the reverse process occurs at rate \(1 - q \neq q\) and vice versa, detailed balance is manifestly broken, \(K^* \neq 0\), and we are dealing with an irreversible NESS. However, we now show that the uniform measure (8) is nevertheless invariant. To see this, consider the total transition rates for all processes leading into or out of a given configuration \(\eta\). We have

\[
\Gamma_{\text{tot}}^{\text{in}}(\eta) = \sum_{\eta'} \Gamma(\eta' \rightarrow \eta) = qN_{\circ \bullet}(\eta) + (1 - q)N_{\bullet \circ}(\eta)
\]

where \(N_{\circ \bullet}(\eta)\) denotes the number of pairs of sites with a particle to the right of a vacancy in the configuration \(\eta\). Similarly

\[
\Gamma_{\text{tot}}^{\text{out}}(\eta) = \sum_{\eta'} \Gamma(\eta' \rightarrow \eta) = qN_{\bullet \circ}(\eta) + (1 - q)N_{\circ \bullet}(\eta).
\]

A little thought reveals that \(N_{\circ \bullet}(\eta) = N_{\bullet \circ}(\eta)\) for any configuration \(\eta\). Hence \(\Gamma_{\text{tot}}^{\text{in}}(\eta) = \Gamma_{\text{tot}}^{\text{out}}(\eta)\) for any \(q\), and the stationarity condition (7) is satisfied for the uniform measure (8).
A few remarks are in order.

(i) The invariance of the uniform measure (8), and the fact that it is independent of the bias \( q \), relies crucially on the ring geometry. With open boundary conditions allowing for the injection and extraction of particles both the SSEP and the ASEP display nontrivial invariant measures characterized by long-ranged correlations and the possibility of phase transitions \([5, 2]\). For example, for the SSEP with boundary densities \( \rho_L \) at \( x = 1 \) and \( \rho_R \) at \( x = L \) one finds a linear mean density profile, as expected from Fick’s law, but in addition there are long-ranged density-density correlations on the scale \( L \), which take the form \([5, 35]\)

\[
\mathbb{E}(\eta_L \xi L \eta L' \xi L') - \mathbb{E}(\eta_L \xi L) \mathbb{E}(\eta L \xi L') = -\frac{\xi(1-\xi')}{L} (\rho_L - \rho_R)^2.
\]

(ii) The invariant measure of the discrete time TASEP on the ring is a Gibbs measure with repulsive nearest-neighbor interactions between the particles \([42, 29, 31]\). This means that the probability of a configuration can be written as a product of pair probabilities,

\[
P^*_\rho(\eta) \sim \prod_x p_\rho(\eta_x, \eta_{x+1}),
\]

where the limit \( N, L \to \infty \) at fixed density \( \rho = N/L \) is implied and

\[
p_\rho(0, 1) = p_\rho(0, 1) = \frac{1 - \sqrt{1 - 4\pi \rho (1 - \rho)}}{2\pi}, \tag{9}
\]

\[
p_\rho(0, 0) = 1 - \rho - p_\rho(1, 0), \quad p_\rho(1, 1) = \rho - p_\rho(1, 0). \tag{10}
\]

For \( \pi \to 0 \) this reduces to a Bernouilli measure of independent particles (see Sect.3.4), whereas for \( \pi \to 1 \) we have \( p_\rho(1, 0) \to (1 - 1 - 2\rho)/2 \), which implies that \( p_\rho(1, 1) \to 0 \) for \( \rho < 1/2 \) and \( p_\rho(0, 0) \to 0 \) for \( \rho > 1/2 \). At \( \pi = 1 \) and mean density \( \rho = 1/2 \) the measure is concentrated on the two configurations \( \eta_x = \pm[1 + (-1)^x]/2 \).

(iii) The invariance of the uniform measure for the ASEP on the ring is an example of pairwise balance \([32]\), a property that generalizes the detailed balance condition (5) into the form

\[
\Gamma(\eta \to \eta') P^*(\eta) = \Gamma(\eta'' \to \eta) P^*(\eta'').
\]

This means that for each configuration \( \eta' \) contributing to the outflux of probability out of the state \( \eta \) there is a configuration \( \eta'' \) whose influx contribution precisely cancels that outflux. In other words, the terms in the sums on the two sides of (7) cancel pairwise.
3.4. Hydrodynamics

An important goal in the study of stochastic interacting particle systems is to understand how deterministic evolution equations emerge from the stochastic microscopic dynamics on large scales [36, 16, 1]. This is similar to the (much harder) problem of deriving hydrodynamic equations from the Newtonian dynamics of molecules in a gas or a fluid. The mathematical procedure involved in the derivation of macroscopic evolution equations for systems with conserved quantities is therefore referred to as the hydrodynamic limit. Here we give a heuristic sketch of hydrodynamics for the ASEP.

The key input going into the hydrodynamic theory is the relationship between particle density $\rho$ and particle current $J$ in the stationary state. From the definition of the ASEP we have

$$J = q\mathbb{E}[\eta_x(1 - \eta_{x+1})] - (1 - q)\mathbb{E}[\eta_x(1 - \eta_{x-1})]$$

where expectations are taken with respect to the invariant measure. Since all configurations of $N$ particles on the lattice of $L$ sites are equally probable,

$$\mathbb{E}[\eta_x(1 - \eta_{x+1})] = \mathbb{E}[\eta_x(1 - \eta_{x-1})] = \frac{N(L - N)}{L - 1}.$$ 

This is just the probability of finding a filled site next to a vacant site, which is obtained by first placing one out of $N$ particle in one of $L$ sites, and then placing one out $L - N$ vacancies in one of the remaining $L - 1$ sites. We conclude that

$$J = \frac{(2q - 1)\rho(1 - \rho)}{1 - 1/L} \to (2q - 1)\rho(1 - \rho) \text{ for } L \to \infty,$$

where the particle density $\rho = N/L$ is kept fixed. Similarly

$$\mathbb{E}[\eta_x\eta_y] = \frac{N(N - 1)}{L(L - 1)} \to \rho^2 \text{ for } L \to \infty$$

for any pair of sites $x \neq y$. This implies (informally) that the invariant measure on the ring converges to a Bernoulli measure on $\mathbb{Z}$ when $L \to \infty$, i.e. each site is occupied independently with probability $\rho$.

We can now formulate the basic idea of the hydrodynamic limit. Suppose that we start the ASEP at time $t = 0$ from a Bernoulli measure with a slowly varying density $\rho(x,0)$. Here “slowly varying” means that variations occur on a scale $\ell \gg 1$. Since the invariant measure of the ASEP is a Bernoulli measure of constant density, it is plausible that, if $\ell$ is chosen large enough, the evolving measure will remain close to a Bernoulli measure with a time and space dependent density $\rho(x,t)$ at all times; and because the particle density is locally conserved, the evolution equation for $\rho(x,t)$ must be of conservation type,

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} j(x,t) = 0.$$ (11)
In the limit $\ell \to \infty$ we may expect, in the spirit of a law of large numbers, that the local particle current $j(x,t)$ converges to the stationary current associated with the local density $\rho(x,t)$,

$$j(x,t) \to J(\rho(x,t)),$$

such that (11) becomes an autonomous, deterministic hyperbolic conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} J(\rho) = 0 \quad (12)$$

for the density profile $\rho(x,t)$. Equation (12) is known as the Euler equation for the ASEP, because similarly to the Euler equation in fluid mechanics it lacks a second order “viscosity” term $\nu \frac{\partial^2 \rho}{\partial x^2}$. It must be emphasized that such a term does not appear when the hydrodynamic limit is carried out rigorously at fixed $q \neq 1/2$. It is only present in the weakly asymmetric case, which implies that $q \to 1/2$ in the limit $\ell \to \infty$ such that $\ell(q-1/2)$ is kept fixed [25].

The Euler equation (12) has been rigorously established for a wide range of models, including cases in which the invariant measure and the current-density relation $J(\rho)$ are not explicitly known [33]. We conclude this section by a brief discussion of the properties of the nonlinear PDE (12), assuming a general (but convex) current-density relation with $J(0) = J(1) = 0$. This includes in particular the discrete time TASEP for which

$$J(\rho) = \pi p_{\rho}(1,0) = \frac{1}{2}[1 - \sqrt{1 - 4\pi \rho(1 - \rho)}]. \quad (13)$$

(i) **Shock formation.** Equation (12) can be rewritten in the form

$$\frac{\partial \rho}{\partial t} + c(\rho) \frac{\partial \rho}{\partial x} = 0,$$

which is the basis for the solution by the method of characteristics. A characteristic is the trajectory of a patch of density $\rho$ which travels at the *kinematic wave speed*

$$c(\rho) = \frac{dJ}{d\rho}.$$

The convexity of the current-density relation implies that $c(\rho)$ is a decreasing function of the density. As a consequence characteristics collide in regions of increasing initial density, $d\rho(x,0)/dx > 0$, leading to the formation of density discontinuities (*shocks*) in finite time. At this point the description by the PDE (12) breaks down, but the speed $V$ of a shock separating regions of density $\rho_L$ on the left and $\rho_R > \rho_L$ on the right is easily inferred from mass conservation to be given by

$$V = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L}.$$
Note that $V \to c$ for $\rho_L \to \rho_R$. On the microscopic level shocks are represented by the shock measures of the ASEP \cite{8,7}. These are inhomogeneous invariant measures on $\mathbb{Z}$ with

$$\lim_{x \to -\infty} E[\eta_x] = \rho_L, \quad \lim_{x \to \infty} E[\eta_x] = \rho_R, \quad \rho_L < \rho_R.$$ 

The microscopic structure of shocks has been studied in considerable detail \cite{6}.

(ii) Rarefaction waves. If the initial density profile is a step function $\rho(x,0) = \begin{cases} \rho_L : & x < 0 \\ \rho_R : & x > 0 \end{cases}$ with $\rho_L > \rho_R$, a diverging fan of characteristics forms leading to a broadening, self-similar density profile

$$\rho(x,t) = \begin{cases} 1 : & x < c(\rho_L)t \\ 0 : & x > c(\rho_R)t \\ \phi(x/t) : & c(\rho_L) < x/t < c(\rho_R), \end{cases} \tag{14}$$

where the shape function $\phi(\xi)$ can be computed from the current-density relation $J(\rho)$. Inserting the ansatz (14) into (12) we see that

$$\phi(\xi) = c^{-1}(\xi).$$

For the continuous time ASEP the interpolating shape is linear, $\phi(\xi) = (1-\xi)/2$.

4. The KPZ conjecture

Much of the work on exclusion processes over the last two decades has been motivated by their connection to surface growth models and problems associated with directed paths (or “polymers”) in random media. On the level of the discrete stochastic process the mapping to a growth model was probably first formulated by Rost \cite{27}, and the directed polymer problem is essentially a re-interpretation of the waiting time representation of the exclusion process \cite{23}. The seminal paper of Kardar, Parisi and Zhang (KPZ) brought this group of problems to the forefront of research in nonequilibrium statistical physics \cite{14}. Working in the framework of a phenomenological stochastic continuum description, they formulated what may be called a universality hypothesis encompassing the fluctuation properties of a large class of different microscopic models. The classic period of research in this area has been extensively reviewed in the literature \cite{22,11,19}. Here we aim to give a concise and simple presentation of the KPZ conjecture, in order to place the more recent developments (to be elaborated in the following sections) into their proper context.
We start from the hydrodynamic equation (12) with a general current-density relation \( J(\rho) \). Since we are interested in fluctuations around a state of constant mean density \( \bar{\rho} \), we write \( \rho(x,t) = \bar{\rho} + u(x,t) \) and expand to second order in \( u \), which yields

\[
\frac{\partial u}{\partial t} = -c(\bar{\rho}) \frac{\partial u}{\partial x} - \lambda u \frac{\partial u}{\partial x},
\]

where

\[
\lambda = \frac{d^2 J}{d^2 \rho}(\bar{\rho}).
\]

The linear drift term on the right hand side can be eliminated by a Galilei transformation \( x \rightarrow x - ct \), which leaves us with what is known (for \( \lambda = 1 \)) as the inviscid Burger equation.

Now fluctuations are introduced (in the spirit of fluctuating hydrodynamics [36]) by adding a random force to the right hand side of (15). In order to guarantee mass conservation, this term must take the form of a derivative \(-\partial \zeta / \partial x\) of a stochastic process \( \zeta(x,t) \) in space and time. This is assumed to be a stationary Gaussian process with zero mean and a covariance function

\[
\mathbb{E}[\zeta(x,t)\zeta(x',t')] = a_x^{-1}a_t^{-1}G[(x-x')/a_x, (t-t')/a_t]
\]

which vanishes beyond a small correlation length \( a_x \) and a short correlation time \( a_t \). Usually one takes formally \( a_x, a_t \rightarrow 0 \), which reduces the right hand side of (17) to a product of \( \delta \)-functions,

\[
\mathbb{E}[\zeta(x,t)\zeta(x',t')] \rightarrow D\delta(x-x')\delta(t-t')
\]

and turns the process \( \zeta(x,t) \) into white noise. This rather violent driving has to be compensated by a small viscosity term \( \nu \partial^2 u / \partial x^2 \) with \( \nu > 0 \). Putting all ingredients together we thus arrive at the stochastic Burgers equation

\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - \lambda u \frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} \equiv -\frac{\partial}{\partial x} j(x,t),
\]

first introduced in the context of randomly stirred fluids [9] and subsequently applied to fluctuations in the exclusion process by van Beijeren, Kutner and Spohn [38]. Here \( j(x,t) \) stands for the local current fluctuation (recall that the macroscopic mean current has been subtracted).

To establish the connection to growth models we introduce the height fluctuation function \( h(x,t) \) through the time-integrated particle current fluctuation,

\[
h(x,t) = \int_0^t j(x,s) ds,
\]

Supplementing this with the initial condition \( u(x,0) = 0 \), it follows from the conservation law for \( u \) that \( \partial h / \partial x = -u \), and therefore

\[
\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \zeta,
\]
which is precisely the KPZ-equation [14].

To proceed, it is important to clearly understand the relation between the stochastic PDE’s (19,21) and the underlying discrete particle systems. The coefficient $\lambda$ in (21) is defined through the current-density relation according to (16), but the viscosity $\nu$ and the noise strength $D$ in (18) do not directly appear on the discrete level. To give these coefficients a consistent interpretation, we start from the observation [9, 12] that the invariant measure of (19) with white-noise driving is white noise with strength $D/2\nu$. This is easy to check for the linearized equation ($\lambda = 0$) but it remains true also for $\lambda \neq 0$, somewhat analogous to the invariance of the uniform measure for the ASEP discussed in Sect.3.3. As a consequence, the spatial statistics of $h(x, t)$ for long times is that of a Wiener process with “diffusion constant” $D/2\nu$ in space,

$$
\lim_{t \to \infty} \mathbb{E}[(h(x, t) - h(x', t))^2] = \frac{D}{2\nu} |x - x'| \equiv \kappa|x - x'|.
$$

This relation holds also on the discrete level, provided $|x - x'|$ is large compared to the correlation length of the particle system, and it identifies the ratio $\kappa = D/2\nu$ as a property of the invariant measure of the latter; for the continuous time ASEP $\kappa = \tilde{\rho}(1 - \tilde{\rho})$ and for the discrete time TASEP $\kappa$ can be computed from the transition probabilities (9). On the particle level the height difference correlation function defined in (22) can be regarded as a measure of the fluctuations in the particle number in the interval between $x$ and $x'$. For this reason $\kappa$ has been referred to as a (nonequilibrium) compressibility [10].

These considerations suggest that the details of the underlying particle system enter the large scale fluctuations properties only through the two parameters $\lambda$ and $\kappa$. These parameters define characteristic scales of height, length and time, which can be used to non-dimensionalize any correlation function of interest. In the non-dimensional variables the correlation functions are then universal, i.e. independent of the specific microscopic model. This is the essence of the universality hypothesis. As an illustration, consider the probability distribution of the height fluctuation $h(x, t)$ at a given point $x$. Because of translational invariance, this cannot depend on $x$, and since the unique combination of $\lambda$, $\kappa$ and $t$ that has the dimension of $h$ is $(\kappa^2 \lambda t)^{1/3}$, we expect that the rescaled height fluctuation

$$
\tilde{h} = \frac{h}{(\kappa^2 \lambda t)^{1/3}}
$$

should have a universal distribution.

Early numerical evidence for universality that goes beyond the observation of $t^{1/3}$-scaling was presented in [22], where it was also pointed out that different universality classes characterized by the same scaling may arise from different initial and boundary conditions on $h$. In the past few years our understanding of this issue has deepened.
considerably thanks to the beautiful mathematical developments which will be described in the following sections.

5. An exactly solvable model: discrete TASEP with step initial conditions

In this section we formulate in theorem 5.4 a result of Johansson [13] on the fluctuations of the particle flux for the discrete TASEP model (discrete in time and space) with step initial data. Somewhat surprisingly these fluctuations are exactly the same, after appropriate rescaling, as the fluctuations of the largest eigenvalue of the Gaussian Unitary Ensemble GUE. We close this section by mentioning a few related results and by a brief overview of Johansson’s proof of theorem 5.4. A more detailed outline of the combinatorics and of the asymptotic analysis used in the proof will be presented in the subsequent sections 6 and 7.

Let us first recall the discrete TASEP model that has been already introduced in (iii) of subsection 3.1. We denote the infinitely many particles of the system by integers \( j = 0, 1, 2, \ldots \) and their respective positions at integer times \( t = 0, 1, 2, \ldots \) by \( x_j(t) \in \mathbb{Z} \). We assume step initial conditions \( x_j(0) = -j \). Jumps to the right \( x_j(t + 1) = x_j(t) + 1 \) are attempted at every time step \( t \geq 0 \) by all particles \( j \geq 0 \) independently, but have to be discarded by the exclusion property if the receiving site is occupied by another particle of the system. In this case, particle \( j \) remains on its site, \( x_j(t + 1) = x_j(t) \).

**Definition 5.1** We denote by \( P_\pi \) the probability measure on the (total) motion of the particle system that is induced by the stochastic process described above.

We give an example on how to compute \( P_\pi \) and determine the probability that the motion depicted in figure 1 occurs. To do this we only need to count for each particle \( j = 0, 1, 2, 3 \) how many times it had a choice to jump and how often it actually jumped.

<table>
<thead>
<tr>
<th>( j )</th>
<th># choices</th>
<th># jumps</th>
<th># stays</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>total</td>
<td>33</td>
<td>16</td>
<td>17</td>
</tr>
</tbody>
</table>

By the assumed stochastic independence of all jumps we have

\[
P_\pi \text{ (figure 1 occurs)} = \pi^{16}(1 - \pi)^{17}.
\]

Next we turn to the flux which is the quantity of interest.

**Definition 5.2** For \( r \in \mathbb{Z}, t \in \mathbb{N} \) we denote the total flux through \( r + \frac{1}{2} \), up to time \( t \) by

\[
F_r(t) := \sharp\{j \in \mathbb{N} : x_j(t) > r\} - \sharp\{j \in \mathbb{N} : x_j(0) > r\}
\]
i.e. the total number of particles that have crossed from site $r$ to $r + 1$ during the time interval $[0, t]$.

For example, in the particular situation displayed in figure 1 we have

<table>
<thead>
<tr>
<th>$t$</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{-1}(t)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$F_0(t)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$F_1(t)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

From now on we will only consider the flux $F_0(t)$ through $r = \frac{1}{2}$ in order to keep the presentation as simple as possible.

The first question one may ask concerns the averaged current $J_\pi := \lim_{t \to \infty} \frac{F_0(t)}{t}$. It is a corollary of theorem 5.4 below that with probability 1 the limit $J_\pi$ exists and equals $\frac{1}{2}(1 - \sqrt{1 - \pi})$. This agrees with formula (13) for $\rho = 1/2$. The more refined question
that was answered by Johansson concerns the deviation of the flux from its mean value. Theorem 5.4 states that

(a) fluctuations \( F_0(t) - J_\pi t \) are of order \( t^{1/3} \) for large values of \( t \).

(b) the distribution of the rescaled random variable \( Z(t) = t^{-1/3}(F_0(t) - J_\pi t) \) can be described as \( t \to \infty \) in terms of the famous Tracy–Widom distribution of Random Matrix Theory.

One may think of this result in analogy to the Central Limit Theorem. There one considers independent, identically distributed random variables \( X_i \). The quantities for which we draw the analogy to the fluxes \( F_0(n) \) are the partial sums \( S_n = X_1 + \ldots + X_n \). Under some weak conditions on the distribution of the \( X_i \)'s one has with probability 1 that \( S_n/n \) converges to the expectation \( \mu := \mathbb{E}(X_1) \) for \( n \to \infty \) (law of large numbers) and that the rescaled random variables \( n^{-1/2}(S_n - n\mu) \) tend to a Gaussian distribution (Central Limit Theorem). We recall a few facts about the Tracy–Widom distribution.

**Reminder 5.3 (Tracy–Widom distribution)**

The Gaussian Unitary Ensemble GUE is defined as a sequence \( \mathbb{P}_N \) of Gaussian probability measures on \( N \times N \) Hermitean matrices of the form

\[
d\mathbb{P}_n(M) = \frac{1}{Z_N} e^{-\text{tr}(M^2)} dM
\]

where \( Z_N \) denotes a suitable norming constant. Denote by \( \lambda_1(M) \) the largest eigenvalue of \( M \) which is a random object. The classic result of Tracy and Widom states that for \( s \in \mathbb{R} \)

\[
\mathbb{P}_N \left( \frac{\lambda_1(M) - \sqrt{2N}}{(8N)^{-1/6}} \leq s \right) \to TW_2(s)
\]

as \( N \to \infty \). The distribution function \( TW_2 \) can be expressed explicitly in terms of the Hastings–McLeod solution of the Painleve II equation or, more implicitly, by Fredholm determinants of integral operators with Airy kernel (see subsection 7.2 for more details).

We are now ready to state the theorem of Johansson for the flux \( F_0(t) \).

**Theorem 5.4** Denote \( j := \frac{1}{2}(1 - \sqrt{1 - \pi}) \), \( d := 2^{-4/3}\pi^{1/3}(1 - \pi)^{1/6} \). Then, for \( 0 < \pi < 1 \) and \( s \in \mathbb{R} \) we have

\[
\lim_{t \to \infty} \mathbb{P}_\pi \left( \frac{F_0(t) - jt}{dt^{1/3}} \leq s \right) = 1 - TW_2(-s)
\]

The results of Johansson in [13] were more general than stated here. They include the description of the fluctuations through other points than \( r = 1/2 \). Moreover, he considered also the continuous (in time) TASEP that is obtained by letting \( \pi \) tend to 0 and by rescaling time in an appropriate manner. We refer the reader to the recent review of [28] for a list of similar results on related models. One striking result is that
changing the initial conditions by requiring that at time zero the particles occupy just
the even sites leads to GOE statistics for the fluctuations of the flux. Furthermore, there
exist results on the spatial correlations of the flux.

In order to explain Johansson’s proof we relate the flux $F_0(t)$ with yet another
random variable. For $j, k \in \mathbb{N}$ denote

$$T(j, k) := \min\{t \in \mathbb{N} : x_j(t) = k + 1 - j\},$$

that is the time by which particle $j$, that starts at site $x_j(0) = -j$, has just completed
its $(k + 1)$-st jump. Observe, that at time $T_k := T(k, k)$ we have

$$x_0(T_k) > x_1(T_k) > \ldots > x_k(T_k) = 1 > 0 \geq x_{k+1}(T_k) > \ldots .$$

Thus, at time $T_k$ exactly the first $k + 1$ particles $0, 1, \ldots, k$ have jumped from site 0 to
site 1 and $F_0(T_k) = k + 1$. Moreover, for times $t < T_k$ we have $F_0(t) \leq k$. This implies
the relation

$$\mathbb{P}_\pi(F_0(t) \leq k) = \mathbb{P}_\pi(T_k > t) = 1 - \mathbb{P}_\pi(T(k, k) \leq t). \quad (24)$$

In the next section we outline how the explicit formula of lemma 6.3 for the
probability distribution of $T(k, k)$ can be derived. By a series of bijections we map
our combinatorial model via waiting times and random words to Semi Standard Young
Tableaux, a classic object of combinatorics and representation theory where explicit
formulas for counting are available. The asymptotic analysis of $\mathbb{P}_\pi(T(k, k) \leq t)$ is
discussed in section 7. The key observation is that the right hand side of (36) is
structurally the same as the standard formula for the probability distribution of the
largest eigenvalue of GUE and the method of orthogonal polynomials can be applied.
The role played by Hermite polynomials for GUE will be taken by Meixner polynomials
in our model. In both cases it is the behavior of the orthogonal polynomials of large
degree in a vicinity of their respective largest zero that matters in the asymptotic
analysis. This behavior can be described with the help of the Airy function for both
Hermite and Meixner polynomials. On a technical level this explains the occurrence
of the Tracy – Widom distribution $TW_2$ for GUE as well as for discrete TASEP with step
initial conditions. We include in section 7 a brief discussion of the universal behavior of
orthogonal polynomials.

6. Proof of theorem 5.4 – part I: Combinatorics

6.1. From discrete TASEP to waiting times

We introduce an equivalent description of the dynamics of the particle system by a table
of waiting times. For $j, l \in \mathbb{N}$ we denote

$$w_{j,l} := \text{number of times particle } j \text{ has decided to stay on site } l - j$$

$$\text{after it became possible to jump to site } l - j + 1.$$
For example, in the case of figure 1 the topleft section of matrix \((w_{j,l})\) reads

\[
\begin{pmatrix}
1 & 2 & 1 & 1 & \ldots \\
1 & 2 & 0 & 2 & \ldots \\
1 & 1 & 0 & 1 & \ldots \\
3 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(25)

The key observation for computing \(T(k, k)\) from the table of waiting times is the following recursion for \(T(j, k)\).

\[
T(j, k) = 1 + w_{j,k} + \begin{cases}
0, & \text{if } j = k = 0 \\
T(j, k - 1), & \text{if } j = 0, k > 0 \\
T(j - 1, k), & \text{if } j > 0, k = 0 \\
\max(T(j - 1, k), T(j, k - 1)), & \text{if } j, k > 0
\end{cases}
\]  

(26)

Indeed, to compute the time it takes the \(j\)-th particle to complete its \((k + 1)\)-st jump one needs to add \(1 + w_{j,k}\) to the time when this jump became possible. For this jump to become possible, particle \(j\) has the be on site \(k - j\) (happens at \(T(j, k - 1)\)) and particle \(j - 1\) must have emptied neighboring site \(k - j + 1\) (happens at time \(T(j - 1, k)\)).

Relation (26) allows to prove the following formula for \(T(j, k)\) by induction on \((j + k)\):

\[
T(j, k) = j + k + 1 + \max_{\mathcal{P} \in \Pi_{j,k}} \left( \sum_{s \text{ on } \mathcal{P}} w_s \right).
\]  

(27)

Here \(\Pi_{j,k}\) denotes the set of paths \(\mathcal{P}\) in the waiting table that connect the \((0,0)\)-entry with the \((j,k)\)-entry and satisfy the additional condition that only steps to the right-neighbor and to the neighbor downstairs are permitted. More formally we may write

\[
\Pi_{j,k} = \left\{ (s_0, \ldots, s_{j+k}) \in (\mathbb{N} \times \mathbb{N})^{j+k+1} : s_0 = (0,0), s_{j+k} = (j,k) \text{ and } s_i - s_{i-1} \in \{(1,0), (0,1)\} \text{ for all } 1 \leq i \leq j+k \right\}.
\]

For \(\mathcal{P} = (s_0, \ldots, s_{j+k}) \in \Pi_{j,k}\) we understand

\[
\sum_{s \text{ on } \mathcal{P}} w_s := \sum_{i=0}^{j+k} w_{s_i}
\]

We illustrate formula (27) with our running example. The corresponding table of waiting times displayed in (25) has two paths in \(\Pi_{3,3}\) that maximize the sum of waiting times. They are

\[
\mathcal{P}_1 : (0,0) \rightarrow (0,1) \rightarrow (0,2) \rightarrow (0,3) \rightarrow (1,3) \rightarrow (2,3) \rightarrow (3,3)
\]

\[
\mathcal{P}_2 : (0,0) \rightarrow (0,1) \rightarrow (1,1) \rightarrow (1,2) \rightarrow (1,3) \rightarrow (2,3) \rightarrow (3,3)
\]
and we have
\[ \sum_{s \text{ on } P_1} w_s = \sum_{s \text{ on } P_2} w_s = 8. \]

Formula (27) then yields \( T(3, 3) = 3 + 3 + 1 + 8 = 15 \) which is easily verified from figure 1.

**Remark 6.1** The probabilistic model we have arrived at, i.e. to search for right- and downward paths that maximize the total waiting time, is also known as the last passage percolation problem and that is precisely the model studied in the paper [13] of Johansson. Interpreting \( w_{j,l} \) as potential energies this can also be considered as the problem of zero-temperature directed polymers in a random medium.

Using the representation of the dynamics of particles by waiting times we obtain the following formula
\[ \mathbb{P}_\pi(T(k, k) \leq t) = \sum_{Q \in W(k,t)} \pi^{(k+1)^2}(1 - \pi)^{|Q|_1}, \] (28)

where \( W(k, t) \) denotes the set of \((k + 1) \times (k + 1)\) matrices \((w_{j,l})\) with entries that are non-negative integers and with the property that
\[ \max_{P \in \Pi_{k,k}} \left( \sum_{s \text{ on } P} w_s \right) \leq t - 2k - 1. \] (29)

For any \( Q \in W(k, t) \) we write \( |Q|_1 \) for the sum of all entries of \( Q \). In order to combinatorially understand the set \( W(k, t) \) we introduce the next transformation.

**6.2. From waiting times to random words**

We associate with any \((k + 1) \times (k + 1)\) matrix \( Q = (w_{j,l}) \) of waiting times the sequence of indices \((j, l)_{0 \leq j, l \leq k}\), listed in lexicographical order, where the value of \( w_{j,l} \) determines how often the index \((j, l)\) appears in this list. In the case of \( Q \) being the principal \(4 \times 4\) submatrix in (25) the corresponding sequence of random words reads
\[
\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
0 & 1 & 1 & 2 & 3 & 0 & 1 & 1 & 3 & 3 & 0 & 1 & 3 & 0 & 0 & 0 & 2
\end{array}
\] (30)

A little thought shows that the quantity \( \max_{P \in \Pi_{k,k}} (\sum_{s \text{ on } P} w_s) \) is encoded in the corresponding sequence of random words as the length of the longest subsequence that is weakly increasing in its second row. The sequence (30) has two such subsequences of maximal length 8, namely
\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 1 & 1 & 2 & 3 & 3 & 3 & 3
\end{array}
\quad \text{and} \quad
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\
0 & 1 & 1 & 1 & 1 & 3 & 3 & 3
\end{array}
\]
that correspond to the maximizing paths $\mathcal{P}_1$ and $\mathcal{P}_2$ introduced above. Formula (28) now reads

$$P_\pi(T(k,k) \leq t) = \sum_{\phi \in D(k,t)} \pi^{(k+1)^2}(1-\pi)^{\text{length of } \phi},$$

where $D(k,t)$ is the set of finite sequences $\phi$ of lexicographically ordered words each consisting of two letters from the alphabet \{0, 1, \ldots, k\} and for which the length of the longest subsequence of $\phi$ with weakly increasing second letter is at most $t - 2k - 1$. By the Robinson–Schensted–Knuth correspondence we may enumerate the set $D(k,t)$ conveniently in terms of Semi Standard Young Tableaux. This is the content of the next subsection.

6.3. From random words to Semi Standard Young Tableaux

The Robinson–Schensted correspondence provides a bijection between permutations and Standard Young Tableaux that is well known in combinatorics and in the representation theory of the permutation group. We now describe the extension of this algorithm to random words which was introduced by Knuth [17]. The basic algorithm that needs to be understood is the row insertion process. Suppose we have a weakly increasing sequence of integers, e.g. 0 0 1 1 1 3. We insert an integer $r$ into this row by the following set of rules. If $r \geq 3$ we simply append $r$ at the end of the row. In the case $r < 3$ we replace the unique number $s$ in the row that is larger than $r$ such that after the replacement the sequence is still weakly increasing. We say that we have inserted $r$ by bumping $s$. For the sequence 0 0 1 1 1 3 insertion of $r$ leads to

<table>
<thead>
<tr>
<th>$r$</th>
<th>sequence after insertion of $r$</th>
<th>bumped number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0 1 1 3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0 0 1 1 1 1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0 0 1 1 1 2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0 0 1 1 1 3 3</td>
<td>no number bumped</td>
</tr>
</tbody>
</table>

A sequence of random words with two letters is transformed into a tableaux of numbers by the following procedure. One inserts the second letters of the words, one after the other, into the first row of the tableaux that has been created before. In case a number is bumped, the bumped number is inserted into the second row of the tableaux. In case bumping occurs again, we insert the newly bumped number into the third row of the tableaux. This process is repeated until the bumping ends. In our example (30) this process leads to the following sequence of tableaux.

$$0 \xrightarrow{1} 0 1 \xrightarrow{1} 0 1 1 \xrightarrow{2} 0 1 1 2 \xrightarrow{3} 0 1 1 2 3 \xrightarrow{0}$$

$$0 0 1 2 3 \xrightarrow{1} 0 0 1 1 3 \xrightarrow{1} 0 0 1 1 1 3 \xrightarrow{3}$$

$$1 \quad 1 2 \quad 1 2 3 \quad 1 2 3$$
Using this procedure we have obtained a Semi Standard Young Tableau with 17 entries.

**Definition 6.2** By a Semi Standard Young Tableaux (SSYT) we understand a tableau $T$ of a finite number of integers that are weakly increasing in each row and strictly increasing in each column. The shape $\lambda = \text{sh}(T)$ of $T$ is denoted by the sequence of row lengths $(\lambda_0, \lambda_1, \ldots)$ that is required to be a weakly decreasing sequence of non-negative integers. Furthermore, we set $|\lambda| := \sum_i \lambda_i$ to be the total number of cells in the tableau.

Observe that we have obtained a list of 17 SSYT’s by the above procedure. The final SSYT $T^*$ has shape $(8, 6, 3, 0, 0, \ldots)$ and the length of the first row equals exactly the length of the longest weakly increasing subsequence. Observe that the sequence (30) is not the only one that leads to the final tableau $T^*$. For example, the sequence

$$0 0 0 0 0 0 1 1 1 1 1 2 2 2 3 3 3 3$$

leads to the same $T^*$. In order to encode our sequence of random words (30) in a one to one way we also have to remember how the tableau grows. The entries of the following tableau encode for our running example at which step which cell has been added to the SSYT.

$$1 2 3 4 5 9 10 13$$

$$6 7 8 12 16 17$$

$$11 14 15$$

Since we also need to remember the first letters of our 17 random words we now replace the entries in (32) in the following way. We note that the first 5 words in (30) have first letter 0 and we therefore replace 1, 2, 3, 4, 5 each by 0. The next five words have first letter 1 and we replace 6, 7, 8, 9, 10 each by 1. Then there are three words starting...
Interacting particle systems out of equilibrium

with letter 2, leading us to replace 11, 12, 13 each by 2. The remaining four entries 14,
15, 16, 17 are each replaced by 3. This leads to another SSYT that clearly has the same
shape as $T^*$. \begin{equation}
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 3 & 3 \\
2 & 3 & 3 \\
\end{array}
\end{equation}

In summary we have described a map that assigns to the sequence (30) of random words
the pair of SSYT’s

\begin{align*}
&\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 2 & 3 \\
1 & 1 & 1 & 1 & 1 & 3 \\
2 & 3 & 3 \\
\end{array}, \\
&\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 3 & 3 \\
2 & 3 & 3 \\
\end{array}
\end{align*}

of equal shape $\lambda$. Clearly, $|\lambda|$ is given by the length of the sequence (30). It is an
instructive exercise to reconstruct the sequence of random words from the pair of SSYT’s
together with a weakly increasing subsequence of the maximal length 8. More generally,
one can show that the set $D(k, t)$ (cf. (31)) is bijectively mapped by the above described
procedure onto pairs $(T, U)$ of SSYT’s of equal shape $\lambda$ satisfying

$$t - 2k - 1 \geq \lambda_0 \geq \lambda_1 \geq \ldots$$

with entries from $\{0, 1, \ldots, k\}$. Note that we have $\lambda_{k+1} = 0$ because entries in each
column are strictly increasing. We therefore arrive at

$$\mathbb{P}_\pi(T(k, k) \leq t) = \sum_{t - 2k - 1 \geq \lambda_0 \geq \ldots \geq \lambda_k \geq 0} \pi^{(k+1)}(1 - \pi)^{\sum_i \lambda_i} L(\lambda, k)^2,$$

where $L(\lambda, k)$ denotes the number of SSYT’s of shape $\lambda = (\lambda_0, \ldots, \lambda_k, 0, \ldots)$ and with
entries from $\{0, 1, \ldots, k\}$. We have now derived a representation for $\mathbb{P}_\pi(T(k, k) \leq t)$
that involves a combinatorial quantity $L(\lambda, k)$ that can be computed explicitly.

6.4. Schur polynomials and an explicit formula for the distribution of $T(k, k)$

There is a beautiful argument using Schur polynomials $s_\lambda$ and classic facts from the
theory of symmetric polynomials that allows to compute $L(\lambda, k)$ explicitly,

$$L(\lambda, k) = \prod_{0 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$ \hspace{1cm} (35)

Introducing the new variables $y_i := \lambda_i - i + k$ and denoting the Vandermonde determinant
by $\Delta(y) = \prod_{0 \leq i < j \leq k}(y_j - y_i)$ we obtain

$$\mathbb{P}_\pi(T(k, k) \leq t) = C_{\pi,k} \sum_{t - 2k - 1 \geq y_0 \geq \ldots \geq y_k \geq 0} \Delta(y)^2 \prod_{i=0}^k (1 - \pi)^{y_i}, \text{ where } (36)$$
\[ C_{\pi,k} := \pi^{(k+1)^2} (1 - \pi)^{-k(k+1)/2} \prod_{0 \leq i < j \leq k} \frac{1}{(j - i)^2} \]  

(37)

Observe that the summand is a symmetric function in \( y \) that vanishes if two components agree. This leads to the final formula in this section.

**Lemma 6.3**

\[ \mathbb{P}_\pi(T(k, k) \leq t) = \frac{C_{\pi,k}}{(k + 1)!} \sum_{y \in \mathbb{Z}^{k+1}} \Delta(y)^2 \prod_{i=0}^{k} (1 - \pi)^{y_i}. \]  

(38)

This formula should be compared with the formula for the distribution of the largest eigenvalue of the Gaussian Unitary Ensemble (cf. reminder 5.3)

\[ \mathbb{P}_N(\lambda_1(M) \leq \Lambda) = \frac{1}{Z_N} \int_{(-\infty, \Lambda]^N} \Delta(y) \prod_{j=1}^{N} e^{-y_j^2} \, dy \]

with some appropriate norming constant \( Z_N \). Observe that this formula has exactly the same structure as (38). The role played by the measure \( e^{-x^2} \, dx \) for GUE is taken by the discrete measure \( \sum_{j=0}^{\infty} (1 - \pi)^j \delta_j \) supported on \( \mathbb{N} \) for discrete TASEP. In the next section we recall how the method of orthogonal polynomials provides a venue to analyze the asymptotics of such types of integrals.

7. **Proof of theorem 5.4 – part II: Asymptotic analysis**

7.1. **The method of orthogonal polynomials following an approach of Tracy and Widom**

We follow the approach of Tracy–Widom [37] to express the right hand side of (38) in terms of Fredholm determinants. Denote for \( x \in \mathbb{Z} \)

\[ w(x) := \begin{cases} 
0 & \text{if } x < 0 \\
(1 - \pi)^x & \text{if } x \geq 0 
\end{cases} \]

Let \( (q_l)_{l \geq 0} \) be any sequence of polynomials with \( q_l \) being of degree \( l \) with (non-zero) leading coefficient \( \gamma_l \). Setting \( \varphi(x) := q_l(x) \sqrt{w(x)} \) we have for \( y \in \mathbb{N}^{k+1} \)

\[ [\det(\varphi(y_i))_{0 \leq i, l \leq k}]^2 = (\gamma_0 \cdots \gamma_k)^2 \Delta(y)^2 \prod_{i=0}^{k} (1 - \pi)^{y_i}. \]

Furthermore, we set \( I_s := [s, \infty) \) and denote by \( 1_{I_s} \), its characteristic function that takes the value 1 on \( I_s \) and 0 on \( \mathbb{R} \setminus I_s \). Using the Leibniz sum for determinants we obtain

\[ \mathbb{P}_\pi(T(k, k) \leq t) = \frac{C_{\pi,k}}{(\gamma_0 \cdots \gamma_k)^2 (k + 1)!} \sum_{y \in \mathbb{Z}^{k+1}} [\det(\varphi(y_i))]^2 \prod_{i=0}^{k} (1 - 1_{I_{t-k}}(y_i)) \]

\[ = \frac{C_{\pi,k}}{(\gamma_0 \cdots \gamma_k)^2} \det S, \]

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where $S$ denotes the $(k + 1) \times (k + 1)$ matrix with entries

$$S_{a,b} = \sum_{x \in \mathbb{Z}} \varphi_a(x) \varphi_b(x) \left(1 - I_{t-k}(x)\right), \quad 0 \leq a, b \leq k.$$ 

So far the choice of the polynomials $q_l$ of degree $l$ was arbitrary. Now we choose $(q_l)_l$ to be the normalized orthogonal polynomials with respect to the discrete measure $\sum_{x \in \mathbb{Z}} w(x) \delta_x$, which belong to the class of Meixner polynomials. We have

$$\sum_{x \in \mathbb{Z}} \varphi_a(x) \varphi_b(x) = \sum_{x \in \mathbb{Z}} q_a(x) q_b(x) w(x) = \delta_{a,b}$$

for $a, b \in \mathbb{N}$. Hence $S = I - R(t - k)$ with

$$R(s)_{a,b} = \sum_{x \in \mathbb{Z}} \varphi_a(x) \varphi_b(x) 1_{I_s}(x) = \sum_{x \geq s} \varphi_a(x) \varphi_b(x).$$

In summary we have so far derived

$$\mathbb{P}_\pi(T(k, k) \leq t) = \det(I - R(t - k)). \quad (39)$$

Observe that one does not need to evaluate the prefactor $C_{\pi,k}(\gamma_0 \ldots \gamma_k)^{-2}$ explicitly to obtain (39). Instead one may deduce that this prefactor must equal 1 by inserting $t = \infty$ in relation (39).

The final idea in the argument of Tracy–Widom is to write $R(s)$ – considered as a linear map $\mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ – as a product $R(s) = A(s)B(s)$, with

$$B(s) : \mathbb{R}^{k+1} \rightarrow \ell_2(\mathbb{Z} \cap I_s), \quad (u_b)_{0 \leq b \leq k} \mapsto \sum_{b=0}^k u_b \varphi_b|_{I_s}$$

$$A(s) : \ell_2(\mathbb{Z} \cap I_s) \rightarrow \mathbb{R}^{k+1}, \quad f \mapsto \left(\sum_{x \geq s} f(x) \varphi_a(x)\right)_{0 \leq a \leq k}$$

Applying the formula $\det(I - AB) = \det(I - BA)$ that holds in great generality we have derived the following Fredholm determinant formula for the probability distribution of $T(k, k)$.

**Lemma 7.1** $\mathbb{P}_\pi(T(k, k) \leq t) = \det(I - \Sigma_k(t - k))$, where

$$\Sigma_k(s) : \ell_2(\mathbb{Z} \cap I_s) \rightarrow \ell_2(\mathbb{Z} \cap I_s), \quad f \mapsto \left(\sum_{y \geq s} \sigma_k(x,y)f(y)\right)_{x \geq s}$$

and $\sigma_k$ denotes the reproducing kernel $\sigma_k(x,y) := \sum_{b=0}^k \varphi_b(x) \varphi_b(y)$ with respect to the Meixner polynomials.

It may seem somewhat strange to convert (39) that involves a determinant of some finite size matrix $I - R$ into a formula that involves the computation of a Fredholm determinant of an operator acting on the infinite dimensional space $\ell_2(\mathbb{Z} \cap I_s)$. However, one has to keep in mind that we are interested in an asymptotic result with $k \rightarrow \infty$. Hence the size...
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of \( I - R \) goes to infinity and it is not at all clear how to perform the asymptotic analysis of the determinants. The operator \( I - \Sigma_k \) on the other hand, acts on the same space for all \( k \) and the dependence on \( k \) is encoded in the reproducing kernels \( \sigma_k \) only. As will be discussed below the kernels \( \sigma_k \) are amenable to asymptotic analysis. In fact, due to the Christoffel–Darboux formula for orthogonal polynomials we may express \( \sigma_k \) just in terms of \( \varphi_k \) and \( \varphi_{k+1} \). For large values of \( k \) the behavior of these functions is rather well understood. For example, if \( x \) is somewhat larger than the largest zero of \( \varphi_k \), then \( |\varphi_k(x)| \) is very close to zero with exponential decay in \( k \). This implies that for values of \( t - k \) that are somewhat larger than the largest zeros of \( \varphi_k \) and \( \varphi_{k+1} \) the operator \( \Sigma_k(t - k) \) is negligible and thus \( \mathbb{P}_\pi(T(k, k) \leq t) \) is very close to 1. If one reduces the value of \( t - k \) to lie in a vicinity of the largest zero of \( \varphi_k \) (which is also close to the largest zero of \( \varphi_{k+1} \)) then the functions \( \varphi_k \) and \( \varphi_{k+1} \), appropriately rescaled, are described to leading order by Airy functions. In the next subsection we will use the just mentioned properties of Meixner polynomials to complete the proof of theorem 5.4.

As it was noted in the last paragraph of subsection 6.4, the formula for the distribution of the largest eigenvalue of GUE is structurally the same as formula (38) for the distribution of \( T(k, k) \) and the arguments described in the present subsection can be applied in an analogous way. The only difference is that we need to use Hermite polynomials rather than Meixner polynomials and that the summation operator \( \Sigma_k \) is to be replaced by an integral operator with a kernel that is given by the reproducing kernel for Hermite polynomials up to degree \( N - 1 \) (\( N \) as in \( \mathbb{P}_N \), cf. reminder 5.3). As in the Meixner case, the leading order behavior of Hermite polynomials near their largest zero is described by the Airy function. On a technical level this is the reason why the fluctuation of the flux in discrete TASEP follows asymptotically the same distribution as the fluctuation of the largest eigenvalue of GUE. It is no coincidence that Meixner polynomials and Hermite polynomials of large degree look the same locally when rescaled appropriately. In fact, large classes of orthogonal polynomials display the same local behavior. We will comment on this universality property of orthogonal polynomials in the last part of the present section.

7.2. Completing the proof of theorem 5.4

We start with a few definitions.

\[
\text{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos(xt + \frac{t^3}{3})dt, \quad x \in \mathbb{R} \quad \text{(Airy function)}
\]

\[
A(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}, \quad x, y \in \mathbb{R} \quad \text{(Airy kernel)}
\]

\[
\text{TW}_2(s) := \det(I - A)_{L^2[s, \infty)}, \quad s \in \mathbb{R} \quad \text{(Tracy–Widom dist. for } \beta = 2),
\]
where in the last definition $A$ denotes the integral operator associated with the Airy kernel. The Tracy–Widom distribution $TW_2$ can be expressed in a more explicit way than by Fredholm determinants of an integral operator. We have

$$TW_2(s) = \exp \left( - \int_s^\infty (x-s)u(x)^2 dx \right),$$

where $u$ denotes the Hastings–McLeod solution of the Painleve II equation $u'' = 2u^3 + xu$ that is singled out from all solutions of this ordinary differential equation by the asymptotic condition $u(x) \sim -\text{Ai}(x)$ for $x \to \infty$. The Airy function in turn solves the linear differential equation $u'' = xu$ with asymptotics $\text{Ai}(x) \sim \frac{x^{-2/3} \exp \left( \frac{-2}{3}x^{3/2} \right)}{2^{1/3} \sqrt{\pi x^{1/4}}} \text{ as } x \to \infty$.

The crucial result in [13] on the reproducing kernel for Meixner polynomials is

$$ck^{1/3} \sigma_k(bk + ck^{1/3} \xi, bk + ck^{1/3} \eta) \to A(\xi, \eta) \text{ for } k \to \infty,$$

(40)

where $b = \pi^{-1}(1 + \sqrt{1-\pi})^2$ and $c = \pi^{-1}(1 - \pi)^{1/6}(1 + \sqrt{1-\pi})^{4/3}$. Together with some technical estimates this implies that the summation operator $\Sigma_k(s)$ introduced in lemma 7.1 is well approximated for large $k$ by the integral operator $A|_{L^2(s^*, \infty)}$ with $s^* = (s - bk)/(ck^{1/3})$. Of course, this approximation is only good for values of $s$ that lie in a neighborhood of $bk$ of size $O(k^{1/3})$. Theorem 5.4 now follows from lemma 7.1 and formula (24).

7.3. Remarks on the universal behavior of orthogonal polynomials

Let $\alpha$ be some measure on $\mathbb{R}$ with positive density $w(x) = d\alpha/dx$ such that all moments $\int x^k w(x) \, dx$ exist and denote by $q_n(x) = \gamma_n x^n + \ldots$ the normalized orthogonal polynomials with respect to $\alpha$, i.e.

$$\int_\mathbb{R} \frac{q_n(x)q_m(x)w(x)}{dx} \, dx = \delta_{n,m}$$

The functions $\varphi_n(x) := q_n(x)\sqrt{w(x)}$ then form an orthonormal system in $L^2(\mathbb{R})$. For many different classes of weights $w$ (and also many discrete measures $\alpha$) the following rough picture arises: The counting measures $\frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)}}$ associated with the zeros $x_i^{(n)}$ of $q_n$ converge for $n \to \infty$ to some measure $\mu$ that is the unique minimizer of

$$I(\mu) = \int \int \log |x-y|^{-1} d\mu(x) d\mu(y) + \int V(x) \, d\mu(x)$$

in the set of measures with total mass 1. The "potential" $V$ depends on $w$. The measure $\mu$ is called the equilibrium measure. The support $J$ of $\mu$ is always contained in the support of $w$. In many cases $J$ is a single interval or a finite collection of intervals. In this situation the behavior of $\varphi_n(x)$ can then be described as follows.

For $x$ outside $J$: $\varphi_n(x)$ decays at an exponential rate to zero as $n \to \infty$.

For $x$ in the interior of $J$: $\varphi_n(x)$ is oscillating rapidly and can be described to leading order by a cosine function with slowly varying frequency and amplitude.
For $x$ close to the boundary of $J$: $\varphi_n(x)$ can be described by special functions. In the case of a soft edge (i.e. the boundary point of $J$ lies in the interior of the support of $w$) the leading order of $\varphi_n(x)$ is generically described by the Airy function.

We conclude this subsection by a few remarks on how to prove the just described asymptotic behavior of $\varphi_n$ for large values of $n$.

I. Differential equations of second order

As an example we discuss Hermite polynomials, i.e. $w(x) = e^{-x^2}$, $x \in \mathbb{R}$. The corresponding functions $\varphi_n$ satisfy the second order differential equations.

$$\varphi''_n(x) + (2n + 1 - x^2)\varphi_n(x) = 0$$

WKB analysis of these differential equations shows that the oscillatory region $|x| < \sqrt{2n+1}$ is connected with the exponential decaying region $|x| > \sqrt{2n+1}$ by Airy functions. This approach can be applied for a number of classic orthogonal polynomials that solve linear differential equations of second order with nice coefficients.

II. Representation by contour integrals

Such representations are known for a number of classic orthogonal polynomials (e.g. for Meixner polynomials) and can be analyzed using the method of steepest descent. Airy functions appear naturally in this context for those values of $x$ where two critical points come close to each other. In such a situation one may generically transform the integral to a normal form near the critical points where the integral representation of the Airy function can be used.

III. Riemann-Hilbert problems

The characterisation of orthogonal polynomials as unique solutions of certain matrix Riemann-Hilbert problems works in principle for all types of weights and opens the way to analyze non-classic orthogonal polynomials. The crucial first step is to solve the variational problem described above for the equilibrium measure $\mu$. In the neighborhood of boundary points of the support of $\mu$ where the density of the equilibrium measure vanishes like a square root (this is the generic case), one obtains by this method Airy functions to describe the asymptotics of the orthogonal polynomials. The method works best in the class of analytic weights, but progress has recently been made for weights that have only a finite number of derivatives.

8. Outlook

Status of universality conjecture; post-Onsager but pre-Wilson era.

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References


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